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Axially symmetric $SU(3)$ gravitating skyrmions

Theodora Ioannidou [a], Burkhard Kleihaus [b], Wojtek Zakrzewski [c]

[a] Maths Division, School of Technology, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
[b] Institut für Physik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany
[c] Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, UK

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Abstract

Axially symmetric gravitating multi-skyrmion configurations are obtained using the harmonic map ansatz introduced in [J. Math. Phys. 40 (1999) 6353]. In particular, the effect of gravity on the energy and baryon densities of the $SU(3)$ non-gravitating multi-skyrmion configurations is studied in detail.

1. Introduction

The Einstein–Skyrme model can be thought of as a non-linear field theory which describes the interaction between a skyrmion (i.e., a baryon) and a black hole. So far, mainly spherically symmetric $SU(N)$ gravitating skyrmions and black hole configurations with Skyrme hair have been obtained [2–4]. Recently, in [5,6] $SU(2)$ static axially symmetric regular and black hole solutions have been derived numerically indicating that the energy densities of such system depend on the coupling constant. In particular, as the coupling constant increases the energy (and baryon) density becomes either denser for the gravitating skyrmions or sparser for the black hole solutions. In what follows we will construct the $SU(3)$ field configurations approximating axially symmetric gravitating skyrmions and black holes (i.e., close to the solutions of the full equations) and investigate their properties.

The $SU(N)$ Einstein–Skyrme action reads:

$$S = \int \left[\frac{R}{16\pi G} + \frac{\kappa^2}{4} \operatorname{tr}\left(K_\mu K^\mu\right) + \frac{1}{32e^2} \operatorname{tr}\left([K_\mu, K_\nu]\left[K^\mu, K^\nu\right]\right) \right] \sqrt{-g}\, d^4x, \tag{1}$$

E-mail addresses: ti3@auth.gr (T. Ioannidou), kleihaus@theorie.physik.uni-oldenburg.de (B. Kleihaus), w.j.zakrzewski@durham.ac.uk (W. Zakrzewski).

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where $K_\mu = \partial_\mu U U^{-1}$ for $\mu = 0, 1, 2, 3$; U is the $SU(N)$ chiral field; g denotes the determinant of the metric; while κ, e are coupling constants and G represents Newton's constant. In order for the finite-energy configurations to exist the Skyrme field has to go to a constant matrix at spatial infinity: $U \to I$ as $|x^\mu| \to \infty$. This effectively compactifies the three-dimensional Euclidean space into S^3 and hence implies that the field configurations of the Skyrme model can be considered as maps from S^3 into $SU(N)$.

The variation of the action (1) with respect to the metric $g^{\mu\nu}$ leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2}$$

with the stress-energy tensor given by

$$T_{\mu\nu} = -\frac{\kappa^2}{2} \operatorname{tr}\left(K_\mu K_\nu - \frac{1}{2} g_{\mu\nu} K_\alpha K^\alpha\right) - \frac{1}{8e^2} \operatorname{tr}\left(g^{\alpha\beta}[K_\mu, K_\alpha][K_\nu, K_\beta] - \frac{1}{4} g_{\mu\nu}[K_\alpha, K_\beta]\left[K^\alpha, K^\beta\right]\right). \tag{3}$$

The starting point of our investigation is the introduction of the coordinates r, z, $\bar{z}$ on $\mathbb{R}^3$. In terms of the usual spherical coordinates r, θ, ϕ the Riemann sphere variable z is given by $z = e^{i\phi} \tan(\theta/2)$. In this system of coordinates the ansatz for the static axially symmetric isotropic metric reads

$$ds^2 = -f\,dt^2 + \frac{m}{f} dr^2 + \frac{r^2}{f}\left(\frac{(m-l)\bar{z}}{z(1+|z|^2)^2} dz^2 + \frac{z(m-l)}{\bar{z}(1+|z|^2)^2} d\bar{z}^2 + \frac{2(m+l)}{(1+|z|^2)^2} dz\,d\bar{z}\right), \tag{4}$$

where the square-root of the determinant of the metric is of the form

$$\sqrt{-g} = \frac{im\sqrt{l}}{f} \frac{2r^2}{(1+|z|^2)^2}, \tag{5}$$

while the metric profiles f, l, m are functions of $(r, |z|^2)$.

The Einstein–Skyrme system has a topological current which is covariantly conserved, yielding the topological charge [7]

$$B = \int \sqrt{-g} B^0\, d^3x, \tag{6}$$

where

$$B^\mu = -\frac{1}{24\pi^2 \sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} \operatorname{tr}(K_\nu K_\alpha K_\beta) \tag{7}$$

and $\varepsilon^{\mu\nu\alpha\beta}$ is the (constant) fully antisymmetric tensor.

In the stereographically projected coordinates the action (1) for the metric (4) becomes equal to

$$\begin{aligned} S = \int dt\,dr\,dz\,d\bar{z}\,\sqrt{-g}\bigg[&\frac{R}{16\pi G} + \frac{\kappa^2}{4} \operatorname{tr}\left(g^{rr} K_r^2 + g^{zz} K_z^2 + g^{\bar{z}\bar{z}} K_{\bar{z}}^2 + 2g^{z\bar{z}} |K_z|^2\right) \\ &+ \frac{1}{16e^2} \operatorname{tr}\left(g^{rr} g^{zz} [K_r, K_z]^2 + g^{rr} g^{\bar{z}\bar{z}} [K_r, K_{\bar{z}}]^2 + \left(g^{\bar{z}\bar{z}} g^{zz} - g^{z\bar{z}\,2}\right)[K_z, K_{\bar{z}}]^2 + 2g^{rr} g^{z\bar{z}} \left|[K_r, K_z]\right|^2\right)\bigg], \end{aligned} \tag{8}$$

while the baryon number takes the form

$$B = -\frac{1}{8\pi^2} \int \operatorname{tr}\left(K_r [K_z, K_{\bar{z}}]\right) dr\,dz\,d\bar{z}. \tag{9}$$

Note that, the metric (4) leads to five non-trivial equations for the three functions f, m and l. In what follows, we consider static fields only (i.e., time independent Skyrme fields). Following the approach of [8] we consider linear superpositions of the stress-energy tensor

$$\mathcal{M}_1 = T_r^r + T_\theta^\theta + T_\phi^\phi - T_0^0, \qquad \mathcal{M}_2 = g_{rr}\big(T_r^r + T_\phi^\phi\big), \qquad \mathcal{M}_3 = g_{rr}\big(T_r^r + T_\theta^\theta\big). \tag{10}$$

Then, the Einstein equations (2) take the form

$$\begin{aligned}
8\pi G\mathcal{M}_1 &= \frac{f_{rr}}{f} + \frac{2}{r}\frac{f_r}{f} - \frac{f_r^2}{f^2} + \frac{1}{2}\frac{f_r}{f}\frac{l_r}{l} + \frac{(1+|z|^2)^2}{r^2}\left[|z|^2\left(\frac{f''}{f} - \frac{f'^2}{f^2}\right) + \frac{1}{2}\frac{f'}{f}\left(1 + |z|^2\frac{l'}{l}\right)\right],\\
8\pi G\mathcal{M}_2 &= \frac{1}{2}\frac{m_{rr}}{m} + \frac{1}{r}\frac{m_r}{m} - \frac{1}{2}\frac{m_r^2}{m^2} + \frac{(3|z|^2-1)|z|^2}{4r^2}\frac{m'}{m} + \frac{(1+|z|^2)^2|z|^2}{2r^2}\left(\frac{m''}{m} - \frac{m'^2}{m^2}\right)\\
&\quad + \frac{1}{2r}\frac{l_r}{l} + \frac{(1+|z|^2)^2|z|^2}{4r^2}\left(2\frac{l''}{l} - \frac{l'^2}{l^2}\right) + \frac{(1+|z|^2)}{2r^2}\frac{l'}{l}\\
&\quad + \frac{1}{4}\frac{m_r}{m}\frac{l_r}{l} + \frac{(1+|z|^2)^2|z|^2}{4r^2}\left(2\frac{f'^2}{f^2} - \frac{m'}{m}\frac{l'}{l}\right),\\
8\pi G\mathcal{M}_3 &= \frac{1}{2}\frac{l_{rr}}{l} + \frac{3}{2r}\frac{l_r}{l} - \frac{1}{4}\frac{l_r^2}{l^2} + \frac{(1+|z|^2)}{2r^2}\frac{l'}{l} + \frac{(1+|z|^2)^2|z|^2}{2r^2}\left(\frac{l''}{l} - \frac{1}{2}\frac{l'^2}{l}\right),
\end{aligned} \tag{11}$$

where l_r denotes the partial derivative of the function l with respect to r and l' the partial derivative with respect to the argument $|z|^2$ (similarly for other functions). Also, the expressions in (10) simplify to

$$\begin{aligned}
\mathcal{M}_1 &= -\frac{1}{16e^2}\frac{f^2(l+m)}{m^2 l}\frac{(1+|z|^2)^2}{r^2}\operatorname{tr}\big|[K_r, K_z]\big|^2 + \frac{1}{32e^2}\frac{f^2}{ml}\frac{(1+|z|^2)^4}{r^4}\operatorname{tr}[K_z, K_{\bar z}]^2,\\
\mathcal{M}_2 &= \frac{\kappa^2}{8}\frac{m}{f}\frac{(1+|z|^2)^2}{|z|^2 r^2}\operatorname{tr}\big(z^2K_z^2 + \bar z^2 K_{\bar z}^2 + 2|z|^2|K_z|^2\big)\\
&\quad + \frac{1}{32e^2}\frac{f}{l}\frac{(1+|z|^2)^2}{|z|^2r^2}\operatorname{tr}\big(z^2[K_r, K_z]^2 + \bar z^2[K_r, K_{\bar z}]^2 - 2|z|^2\big|[K_r, K_z]\big|^2\big),\\
\mathcal{M}_3 &= -\frac{\kappa^2}{8}\frac{m}{l}\frac{(1+|z|^2)^2}{|z|^2 r^2}\operatorname{tr}\big(z^2K_z^2 + \bar z^2 K_{\bar z}^2 - 2|z|^2|K_z|^2\big)\\
&\quad - \frac{1}{32e^2}\frac{f}{m}\frac{(1+|z|^2)^2}{|z|^2r^2}\operatorname{tr}\big(z^2[K_r, K_z]^2 + \bar z^2[K_r, K_{\bar z}]^2 + 2|z|^2\big|[K_r, K_z]\big|^2\big).
\end{aligned} \tag{12}$$

Next we consider the static Einstein–Skyrme equations and construct field configurations approximating their axially symmetric solutions based on the harmonic map ansatz using the formulations of [1].

2. The harmonic map ansatz

The idea of the harmonic map ansatz (i.e., the generalisation of the rational map ansatz of Houghton et al. [9]) involves the separation of the radial and angular dependence of the fields [1] as

$$U = e^{2ih(r)(P - I/N)} = e^{-2ih(r)/N}\big[I + \big(e^{2ih(r)} - 1\big)P\big], \tag{13}$$

where P is a $N \times N$ Hermitian projector which depends only on the angular variables $(z, \bar z)$ and $h(r)$ is the corresponding profile function. Note that the matrix P can be thought of as a mapping from S^2 into CP^{N-1}. Thus, P can be written as

$$P(V) = \frac{V \otimes V^\dagger}{|V|^2}, \tag{14}$$

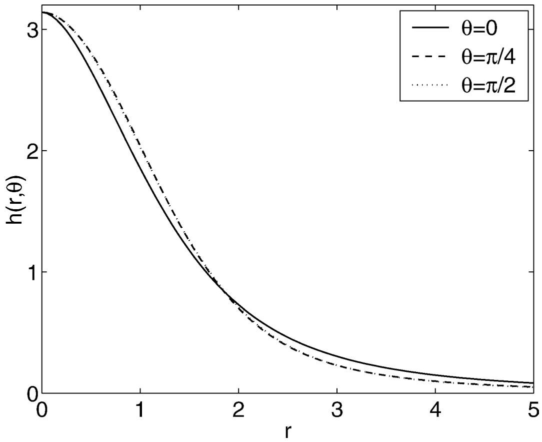

Fig. 1. The $|z|^2$ dependence of the Skyrme field profile function. [Recall, $|z| = \tan\frac{\theta}{2}$].

where V is a N-component complex vector (dependent on z and $\bar{z}$). Following [10], we define a new operator P_+, by its action on any vector $v \in \mathbb{C}^N$, as

$$P_+ v = \partial_z v - \frac{v(v^\dagger \partial_z v)}{|v|^2}, \tag{15}$$

and then define further vectors $P_+^k v$ by induction: $P_+^k v = P_+(P_+^{k-1} v)$. As shown in [11], these vectors have many interesting properties when V is holomorphic. These properties either follow directly from the definition of the P_+ operator or are very easy to prove and they lead to many simplifications of the expressions for the action and the energy–momentum tensor.

For (13) to be well-defined at the origin, the radial profile function $h(r)$ has to satisfy $h(0) = \pi$ while the boundary value $U \to I$ at $r = \infty$ requires that $h(\infty) = 0$. As shown in [1], an attractive feature of the ansatz (13) is that it leads to a simple expression for the energy density which can be successively minimized with respect to the parameters of the projector P and then with respect to the shape of the profile function $h(r)$. This procedure gives good approximations to multi-skyrmion field configurations in flat space [1].

Since we are interested in axially symmetric field configurations in curved space the action possesses, in addition to the explicit $|z|^2$ dependence due to the complex vector v, an implicit $|z|^2$ dependence induced by the metric functions. Consequently, we cannot simply integrate the action over $|z|^2$ (as in flat space). Therefore, we consider a slightly modified ansatz by allowing the profile function h to depend on r and $|z|^2$ (shown in Fig. 1). In fact, this ansatz (concerned with the Skyrme part) involves a generalisation of the rational map ansatz, the so-called improved harmonic map ansatz introduced in [12]. Indeed [12] presents better approximations (i.e., with lower energies) of pure $SU(2)$ skyrmions and so are better approximants of the exact solutions obtained numerically by solving the full equations.

The action (8) after the substitution of (13) with $h = h(r, |z|^2)$ and having used the properties of the harmonic maps becomes

$$\begin{aligned} S = \int \Bigg(& \mathcal{K} r^2 \frac{R}{8\pi G} - \kappa^2 A_N r^2 h_r^2 - \kappa^2 B_N h'^2 - \left(\kappa^2 \mathcal{N}_1 + \frac{1}{e^2} \mathcal{N}_2 h_r^2 + \frac{1}{e^2} \mathcal{N}_3 \frac{h'^2}{r^2} \right) \sin^2 h \\ & - \frac{1}{e^2} \mathcal{I} \frac{\sin^4 h}{r^2} \Bigg) dt\, dr\, dz\, d\bar{z}, \end{aligned} \tag{16}$$

where

$$\mathcal{K}=i\frac{m\sqrt{l}}{f}\frac{1}{(1+|z|^2)^2},\qquad A_N=2i\frac{N-1}{N}\frac{\sqrt{l}}{(1+|z|^2)^2},\qquad B_N=2i\frac{N-1}{N}\sqrt{l}|z|^2,$$
$$\mathcal{N}_1=i\frac{(m+l)}{\sqrt{l}}\frac{|P_+V|^2}{|V|^2},\qquad \mathcal{N}_2=i\frac{f}{m}\frac{(m+l)}{\sqrt{l}}\frac{|P_+V|^2}{|V|^2},\qquad \mathcal{N}_3=i\frac{f}{\sqrt{l}}\frac{|P_+V|^2}{|V|^2}\big(1+|z|^2\big)^2|z|^2,$$
$$\mathcal{I}=i\frac{f}{\sqrt{l}}\frac{|P_+V|^4}{|V|^4}\big(1+|z|^2\big)^2. \tag{17}$$

The baryon number (9) coincides with the expression for the topological charge of the CP^{N-1} sigma model (up to an overall profile dependent factor) since

$$B=\frac{i}{\pi^2}\int \operatorname{tr}\big(P[P_z,P_{\bar z}]\big)\,dz\,d\bar z\int_0^\infty \sin^2 h h_r\,dr=\frac{i}{2\pi}\int\frac{|P_+V|^2}{|V|^2}dz\,d\bar z. \tag{18}$$

The r integration can be performed although $h=h(r,|z|^2)$ since $\int_0^\infty \sin^2 h h_r\,dr=h(0)/2$ and we have required that $h(0,|z|^2)=\pi$.

The variation of (16) with respect to h gives its equation of motion:

$$\partial_x\big[\big(A_N x^2+\mathcal{N}_2\sin^2 h\big)h_x\big]+\bigg[\bigg(B_N+\frac{1}{x^2}\mathcal{N}_3\sin^2 h\bigg)h'\bigg]'$$
$$-\bigg(\frac{1}{2}\mathcal{N}_1+\frac{1}{2}\mathcal{N}_2h_x^2+\frac{1}{2}\mathcal{N}_3\frac{h'^2}{x^2}+\mathcal{I}\frac{\sin^2 h}{x^2}\bigg)\sin(2h)=0, \tag{19}$$

where the dimensionless coordinate $x=e\kappa r$ has been introduced.

In addition, the left-hand side components (12) of the Einstein equations (11) become

$$\mathcal{M}_1=e^2\kappa^4\frac{f^2}{ml}\frac{\sin^2 h}{x^2}\frac{|P_+V|^2}{|V|^2}\big(1+|z|^2\big)^2\bigg[\frac{l+m}{m}h_x^2+\frac{(1+|z|^2)^2}{x^2}\bigg(|z|^2h'^2+\sin^2 h\frac{|P_+V|^2}{|V|^2}\bigg)\bigg],$$
$$\mathcal{M}_2=e^2\kappa^4\big(1+|z|^2\big)^2\bigg[-2|z|^2\frac{N-1}{N}\frac{h'^2}{x^2}+\bigg(-1+\frac{f}{l}h_x^2\bigg)\frac{\sin^2 h}{x^2}\frac{|P_+V|^2}{|V|^2}\bigg],$$
$$\mathcal{M}_3=e^2\kappa^4\bigg(-\frac{m}{l}+\frac{f}{m}h_x^2\bigg)\frac{|P_+V|^2}{|V|^2}\big(1+|z|^2\big)^2\frac{\sin^2 h}{x^2}. \tag{20}$$

In [1,9] the explicit forms of the holomorphic vector V have been obtained by minimizing the $SU(2)$ and $SU(3)$ skyrmion energy densities, respectively. In particular, the low energy field configurations of the $SU(2)$ Skyrme model with baryon number from one up to seventeen and of the $SU(3)$ Skyrme model with baryon number from one up to six, which are not the $SU(2)$ embeddings, have been constructed. From these, the axisymmetric field configurations are the following:

- $SU(2)$ $B=2$ *skyrmion*: the associated vector V has the simple form [9]

$$V=\big(z^2,1\big)^t,\qquad \frac{|P_+V|^2}{|V|^2}=\frac{4|z|^2}{(1+|z|^4)^2}. \tag{21}$$

  Recently, this case has been studied in [5].
- $SU(3)$ $B=4$ *skyrmion*: the harmonic vector V is of the form [1]:

$$V=\big(z^4,az^2,1\big)^t,\qquad \frac{|P_+V|^2}{|V|^2}=\frac{4|z|^2(a^2(1+|z|^8)+4|z|^4)}{(|z|^8+a^2|z|^4+1)^2} \tag{22}$$

for $a = 2.7191$. The corresponding configuration has the shape of *two tori on top of each other close to the equator of the sphere* and its energy and baryon density are invariant under a rotation around the z-axis, which is equivalent to the $SU(3)$ transformation: $U \to A^{-1}UA$ for $A = \mathrm{diag}(e^{-2i\alpha}, 1, e^{-2i\alpha})$.

Next we use the vectors V given by (21) and (22) in order to obtain approximations to the gravitating $SU(2)$ $B = 2$ and $SU(3)$ $B = 4$ skyrmions, respectively.

3. Numerical solutions

Due to convenience of our numerical scheme we transform our expressions back to spherical coordinates where the metric is parametrized in terms of (x, θ, ϕ) as

$$ds^2 = -f\,dt^2 + \frac{m}{f}dx^2 + \frac{m}{f}x^2\,d\theta^2 + \frac{l}{f}r^2\sin^2\theta\,d\phi^2 \tag{23}$$

while the Einstein equations (2) take the form

$$\begin{aligned}
2\alpha M_1 &= \frac{f_{xx}}{f} + \frac{1}{x^2}\frac{f_{\theta\theta}}{f} + \frac{2}{x}\frac{f_x}{f} - \frac{1}{x^2}\frac{f_\theta^2}{f^2} - \frac{f_x^2}{f^2} + \frac{\cot\theta}{x^2}\frac{f_\theta}{f} + \frac{1}{2}\frac{f_x}{f}\frac{l_x}{l} + \frac{1}{2x^2}\frac{f_\theta}{f}\frac{l_\theta}{l},\\
2\alpha M_2 &= \frac{1}{2}\frac{m_{xx}}{m} + \frac{1}{x}\frac{m_x}{m} + \frac{1}{2x^2}\frac{m_{\theta\theta}}{m} - \frac{1}{2x^2}\frac{m_\theta^2}{m^2} - \frac{1}{2}\frac{m_x^2}{m^2} + \frac{1}{2x^2}\frac{l_{\theta\theta}}{l} - \frac{1}{4x^2}\frac{l_\theta^2}{l^2} + \frac{1}{2x^2}\frac{f_\theta^2}{f^2}\\
&\quad + \frac{1}{2x}\frac{l_x}{l} + \frac{1}{2x}\frac{m_x}{m} + \frac{1}{4}\frac{m_x}{m}\frac{l_x}{l} - \frac{1}{4x^2}\frac{m_\theta}{m}\frac{l_\theta}{l} - \frac{\cot\theta}{2x^2}\frac{m_\theta}{m} + \frac{\cot\theta}{x^2}\frac{l_\theta}{l},\\
2\alpha M_3 &= \frac{1}{2}\frac{l_{xx}}{l} + \frac{1}{2x^2}\frac{l_{\theta\theta}}{l} - \frac{1}{4x^2}\frac{l_\theta^2}{l^2} - \frac{1}{4}\frac{l_x^2}{l^2} + \frac{3}{2x}\frac{l_x}{l} + \frac{l_\theta}{l}\cot\theta.
\end{aligned}\tag{24}$$

The dimensionless coupling constants α and M_i are defined as: $\alpha = 4\pi G\kappa^2$ and $M_i = \mathcal{M}_i/e^2\kappa^4$ for $i = 1, 2, 3$.

This system of coupled non-linear partial differential equations has to be solved numerically subject to the given boundary conditions. To map the infinite interval of the radial variable r onto the finite interval $[0, 1]$ we introduce the variable $\bar{x}$ as

$$\bar{x} = \frac{x}{1+x}. \tag{25}$$

Since the functions are symmetric under the transformation $\theta \to \pi - \theta$, it is sufficient to solve the equations for only $0 \leqslant \theta \leqslant \pi/2$ with the boundary conditions

$$h_\theta(\bar{x}, \theta = \pi/2) = f_\theta(\bar{x}, \theta = \pi/2) = m_\theta(\bar{x}, \theta = \pi/2) = l_\theta(\bar{x}, \theta = \pi/2) = 0. \tag{26}$$

To satisfy the regularity condition $m(\bar{x}, \theta = 0) = l(\bar{x}, \theta = 0)$ we introduce the function

$$g(\bar{x}, \theta) = \frac{m(\bar{x}, \theta)}{l(\bar{x}, \theta)} \tag{27}$$

and impose the boundary conditions $g(\bar{x}, \theta = 0) = 1$, $g(\bar{x} = 0, \theta) = 1$, $g(\bar{x} = 1, \theta) = 1$ and $g_\theta(\bar{x}, \theta/\pi/2) = 0$. The numerical computations, based on the Newton–Raphson method, have been performed using the program FIDISOL [13] where the partial differential equations are discretized on a non-equidistant grid with typical grids sizes 130×30 and estimated relative numerical errors of order 10^{-3}.

It has been observed that, for a fixed complex vector V, the globally regular solutions of the Einstein–Skyrme model depend *only* on the dimensionless coupling constant $\alpha = 4\pi G\kappa^2$. In fact, the numerical simulations show that such field configurations exist only for a finite range of α: $0 \leqslant \alpha \leqslant \alpha_{\max}$ where its maximal value depends

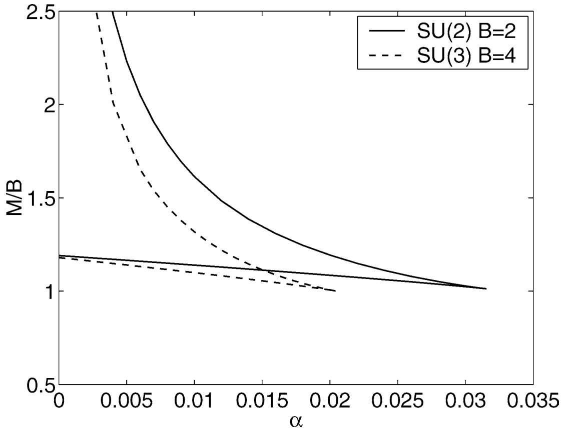

Fig. 2. The dimensionless mass M/B in terms of the coupling constant α in the $SU(2)$ and $SU(3)$ case.

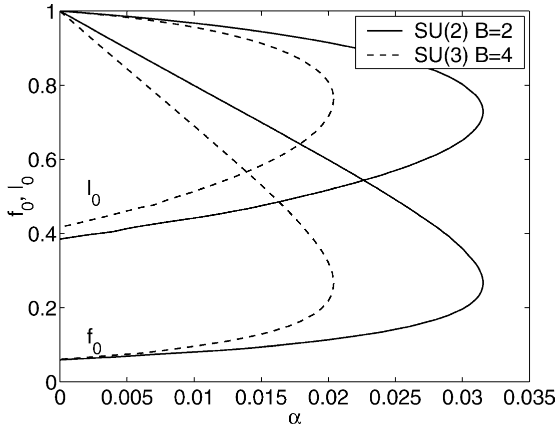

Fig. 3. The axial part of the initial metric profiles $f(0,\theta)$ and $l(0,\theta)$ as functions of the gravitational constant α in the $SU(2)$ and $SU(3)$ case.

on the specific form of V. For instance, in the $SU(2)$ $B=2$ case where V is given by (21) $\alpha_{\max}=0.0338$, while in the $SU(3)$ $B=4$ case where V is given by (22) with $a=2.7191$, $\alpha_{\max}=0.0209$. Assuming that our field configurations are very close to the proper solutions of the field equations (as shown in [12]) we believe that what we have seen here is also true for the exact (skyrmion) solutions.

Hence we note that starting from a skyrmion in flat space a branch of gravitating skyrmions emerges for $\alpha>0$. This branch terminates at the maximal value of the coupling constant $\alpha_{\max}$, where it joints smoothly a second branch of solutions, which bends backwards from $\alpha=\alpha_{\max}$ to $\alpha=0$.

Fig. 2 presents the dimensionless mass $M/B=-(6\pi^2\kappa/e)^{-1}(\int T_0^0\sqrt{|g|}\,d^3r)/B$ as a function of α for the $SU(2)$ $B=2$ and $SU(3)$ $B=4$ skyrmions. The mass decreases along the first branch with increasing α until it reaches its minimum at $\alpha=\alpha_{\max}$ then it increases along the second branch with decreasing α and diverges as α tends to zero. Accordingly, we refer to the branches with lower (respectively, higher) mass as the lower (respectively, upper) branch.

Fig. 3 presents the α dependence of the metric functions at the origin $f_0=f(0,\theta)$ and $l_0=l(0,\theta)$ for the $SU(2)$ $B=2$ and the $SU(3)$ $B=4$ field configurations. Note that, f_0 and l_0 decrease monotonically for the configurations along the lower branch with increasing α and along the upper branch with decreasing α; they approach finite values as α tends to zero on the upper branch.

Figs. 4 and 5 present the dimensionless energy density $-T_0^0$ and the baryon density B^0 as functions of $\rho=x\sin\theta$ and $z=x\cos\theta$ for several values of the coupling constant α for the $SU(2)$ $B=2$ and $SU(3)$ $B=4$ skyrmions. In the $SU(2)$ $B=2$ case, both T_0^0 and B^0 possess a maximum on the ρ-axis which corresponds to a ring in the xy-plane; in accordance with the toroidal symmetry of the $SU(2)$ $B=2$ non-gravitating skyrmion. The radius of the torus decreases with increasing α on the lower branch. On the upper branch it continues to decrease with decreasing α and tends to zero as α tends to zero. The height of the maximum increases on the lower branch with increasing α and on the upper branch with decreasing α and diverges on the upper branch as α tends to zero. On the other hand, the $SU(3)$ $B=4$ gravitating skyrmion has the shape of a double torus as can be observed directly from Fig. 5 (similar to the flat case). Note that, as α increases the radii of the tori and their distance from the xy-plane decrease along the lower branch and shrink to zero along the upper branch as α decreases. Simultaneously, the maxima of T_0^0 and B^0 increase and diverge on the upper branch as α tends to zero. This singular behaviour is seen in our choice of the dimensionless radial coordinate $x=e\kappa r$ as first observed by Bizon and Chmaj [14]. If the rescaled coordinate $\tilde{x}=x/\sqrt{\alpha}$ is used instead, all functions are smooth and the rescaled energy $\tilde{M}/B=\sqrt{\alpha}M/B$ remains finite as α tends to zero on the upper branch.

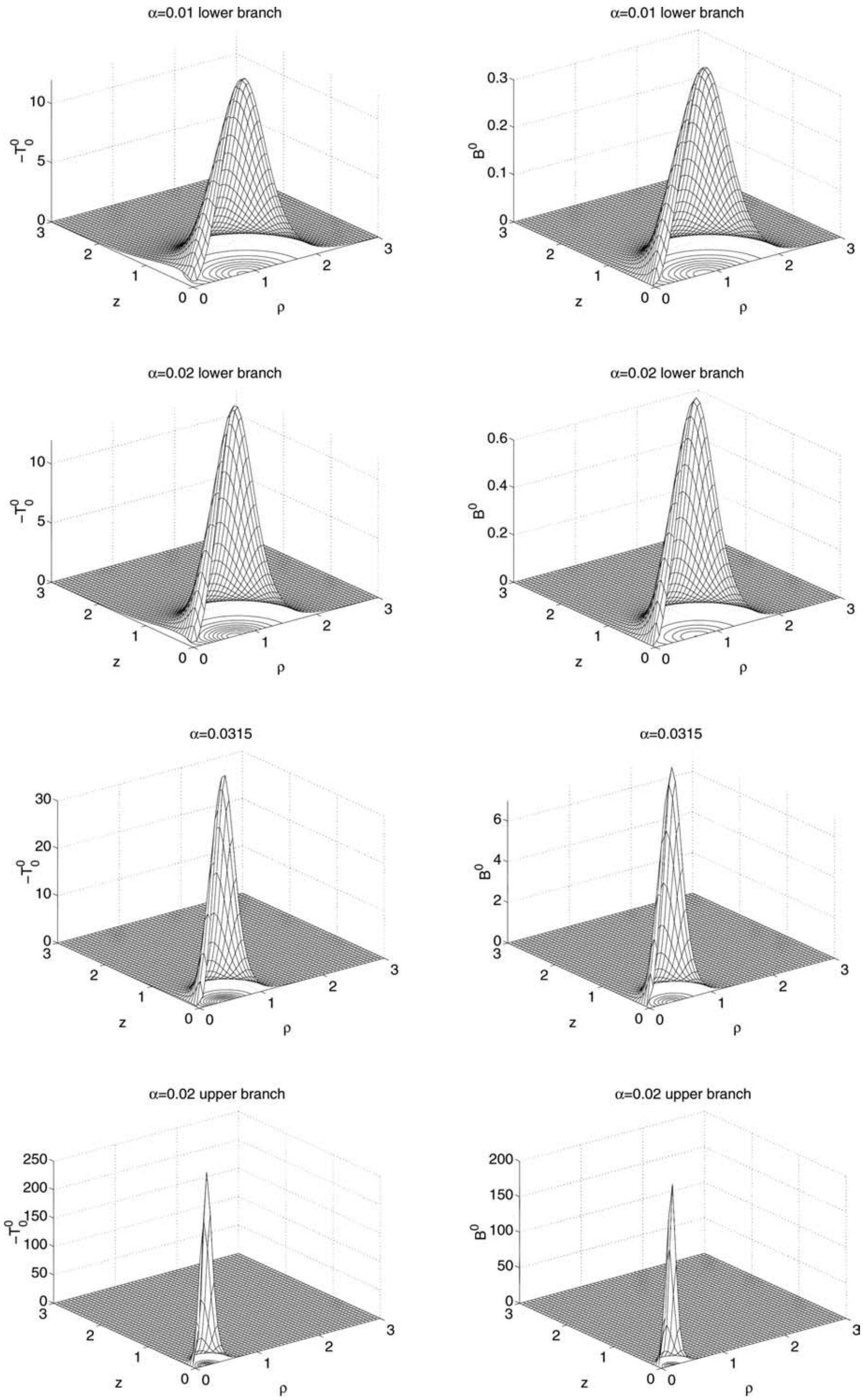

Fig. 4. The energy density $\epsilon = T_0^0$ (left) and the baryon number density B^0 (right) are shown for the $SU(2)$ $B = 2$ solution.

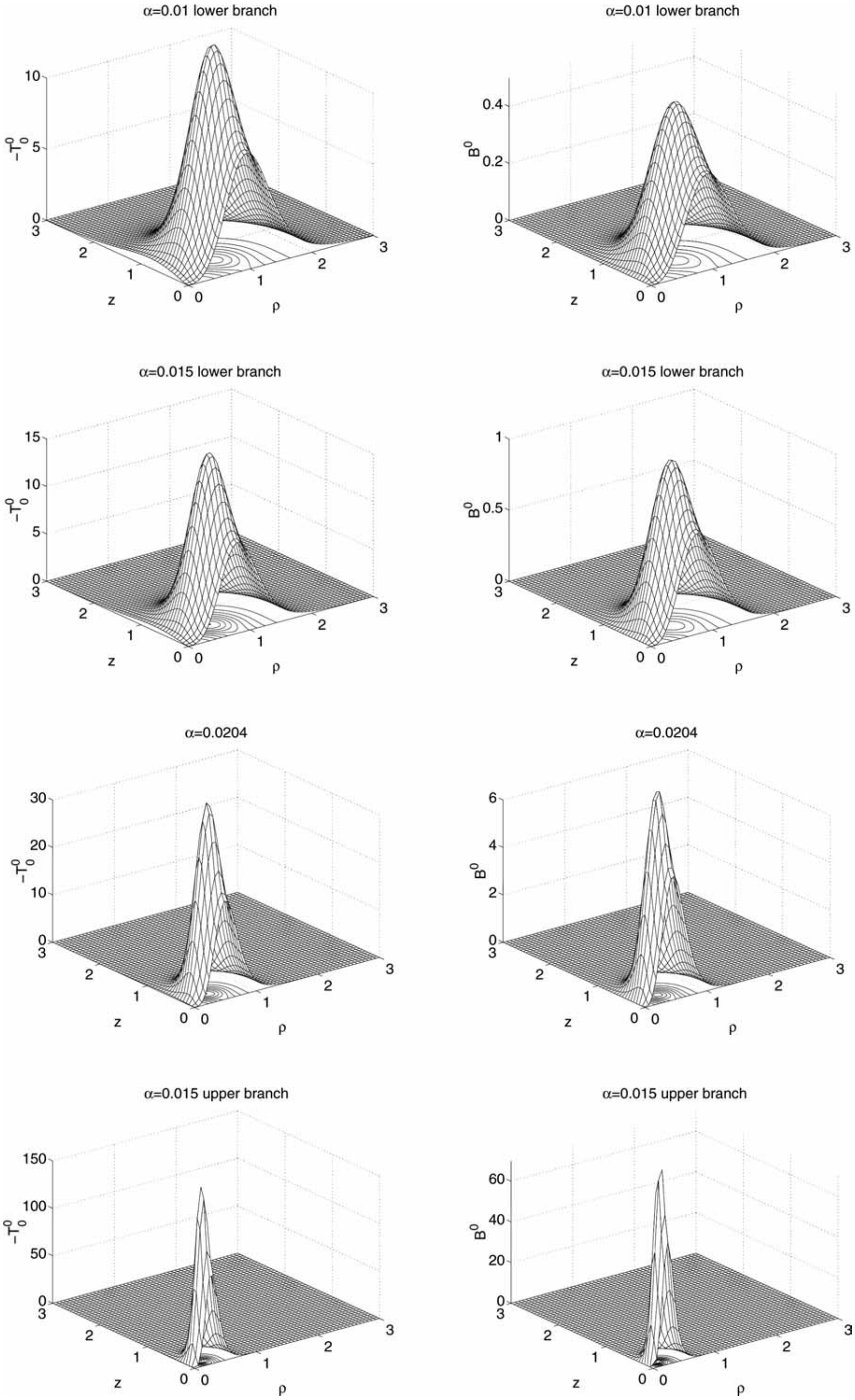

Fig. 5. Similar as Fig. 3 for the $SU(3)$ $B = 4$ solution.

Finally we note that, for the $SU(2)$ $B = 2$ and $SU(3)$ $B = 4$ gravitating skyrmions the dependence on the coupling constant α follows the same pattern as for the spherically symmetric $SU(2)$ and $SU(3)$ skyrmions.

4. Conclusions

In this Letter, we have used the improved harmonic map ansatz to look at field configurations involving gravitating skyrmions. In particular we have derived approximations of the $SU(2)$ $B = 2$ and $SU(3)$ $B = 4$ gravitating skyrmions. Our configurations are not solutions of the equations of motion but we believe that they are close to them; hence we hope that our results are close to what would have been seen for the solutions.

We have found that (qualitatively) the situation is very similar to the four spherical symmetric $SU(3)$ skyrmions [4] since the equations have solutions only for a range of the gravitational coupling constant with the skyrmions being bound by the gravitational field. That is, for values of the coupling constant below its critical value—we have two solutions (of which one has much higher energy) and when the coupling constant goes beyond its critical value—there are not unitary solutions for the Skyrme fields suggesting that the system possesses two complex solutions. The shapes of the energy densities are very similar to what is seen for the non-gravitating skyrmions. Thus we suspect that the effects of the gravitational field are universal in nature; the field binds the skyrmions and it alters their properties in a universal way making them more compact but not changing their basic shape.

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